Shallowly Embedding Type Theories as Presheaf Models in Agda

Extended Abstract

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1 Introduction

There are several type systems which extend Martin-Löf Type Theory (MLTT) by providing extra operations on types and terms or by including new axioms. Of course, the ability to use these new constructs when proving a theorem or when writing a program does not come for free, one has to check that the resulting type system remains consistent. In a lot of cases this can be accomplished by constructing a presheaf model of the type theory in which all the extra operations and axioms are given a semantic meaning [8]. Examples include presheaf models of type theories with support for guarded recursion [4], with support for parametricity [1, 11, 12], and of univalent/cubical type theory [5].

However, in order to use these new features in a proof assistant based on MLTT such as Agda, one basically has two options: either to postulate the new operations or axioms, in which case they will not have any computational content, or to implement an extension of the proof assistant, which has happened for instance with Agda's cubical mode [15] but which requires a lot of work (and in general this effort needs to be repeated for every extension of MLTT one wants to consider). In both approaches, soundness of the extensions needs to be proven separately, meta-theoretically.

In this extended abstract, we present work in progress on a shallow embedding of extensions of MLTT in Agda as presheaf models. More concretely, the terms, types, ... of an object theory are represented in Agda using the presheaf construction and a user can manipulate them in the style of a category with families (CwF) [6, 8], with variables in de Bruijn form. Extra operations or axioms can be implemented by instantiating the framework with a suitable base category.

Most Agda definitions appearing in the text will only include the type, not the implementation. Details such as universe levels will be elided to enhance readability. The full code can be found at https://github.com/JorisCeulemans/shallow-presheaf-embedding/tree/tyde-2020.

2 Overview of the Framework

Our framework is parametrized by a small base category C that will depend on the extension of MLTT under consideration. For such a category C : Category, we denote by Ob C

: Set its type of objects and by Hom C x y : Set the type of morphisms from an object x to an object y.

The framework then consists of Agda (record) types for the different kinds of judgements present in MLTT. The overall structure is that of Dybjer's internal CwFs [6]. This means that we first introduce the notion of contexts, then that of types in a context and finally that of terms of a type in a context.

First of all, a context is represented as a presheaf over the base category C (i.e. a contravariant functor from C to the category of Agda types and functions).

```
record Ctx (C : Category) : Set where
  field
    set : Ob C → Set
    rel : ∀ {x y} → Hom C x y → set y → set x
    (...)
```

Here two fields expressing the functor laws were elided.

Similarly, we have for any context Γ : Ctx C an Agda type of types in this context,

```
record Ty (Γ : Ctx C) : Set where (...)
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and for every type T : Ty Γ there is an Agda type of terms of type T in the context Γ.

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record Tm (Γ : Ctx C) (T : Ty Γ) : Set where (...)
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The precise types of the fields in these records will not be important in the rest of the discussion.

Furthermore, we provide for any two contexts Δ and Γ a type $\Delta \Rightarrow \Gamma$ of substitutions from Δ to Γ and an action of a substitution on types and terms. More concretely, if $\sigma : \Delta \Rightarrow \Gamma$ and T : Ty Γ and t : Tm Γ T, then T [σ] : Ty Δ and t [σ]' : Tm Δ (T [σ]). We can also extend a context Γ with a type T : Ty Γ to obtain a context Γ „ T (which would be written in MLTT as $\Gamma, x : T$) and then we get a substitution $\pi : \Gamma$ „ $T \Rightarrow \Gamma$ and a term ξ : Tm (Γ „ T) (T [π]). This term ξ corresponds to the variable rule in MLTT for the last variable in the context (so to the judgement $\Gamma, x : T \vdash x : T$). Finally, there are Agda types expressing equality of substitutions, of types and of terms.[1] Each of these types will in the text be denoted by $\_\cong\_$. Given an equality proof $e : T \cong S$ for two types T, S : Ty Γ, a term s : Tm Γ S can be converted into a term ι[e] s : Tm Γ T.

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[1] Working with these custom-defined equality types turns out to be easier than with standard propositional equality.

Moreover, in any presheaf model (irrespective of the base category C) we can construct simple types, such as booleans and natural numbers, and some basic type operators and term constructors. For example, we have a type Nat' : Ty Γ of natural numbers and a term zero' : Tm Γ Nat' for any context Γ and we can implement simple product types

_⊠_ : Ty Γ → Ty Γ → Ty Γ
pair : Tm Γ T → Tm Γ S → Tm Γ $(T \boxtimes S)$
fst : Tm Γ $(T \boxtimes S)$ → Tm Γ T
snd : Tm Γ $(T \boxtimes S)$ → Tm Γ S

and simple (non-dependent) function types.

_⇒_ : Ty Γ → Ty Γ → Ty Γ
lam : $(T :$ Ty $\Gamma) \to$ Tm $(\Gamma \,,\, T)$ $(S \,[\, \pi \,]) \to$ Tm Γ $(T \Rightarrow S)$
app : Tm Γ $(T \Rightarrow S) \to$ Tm Γ $T \to$ Tm Γ S

Note that the body of a lambda abstraction has type $S \,[\, \pi \,]$ rather than S because the latter is not a type in context $\Gamma \,,\, T$.

3 A Concrete Example: Guarded Recursion

To demonstrate how we intend to use the embedding of presheaf models in Agda described above, we will consider in this section guarded recursion as a specific application. Guarded recursion was originally developed by Nakano [10], it is a technique for writing productive recursive definitions involving coinductive data types using a modality ▷ on types called "later". Presheaf models for guarded recursion were described in for instance [3, 4].

We can work with guarded recursion in our framework by instantiating it with the category ω as the base category (this is the category structure induced on the set $\mathbb{N}$ of natural numbers by its standard order relation). In this case we can define a later modality ▷' on types.[2]

▷' : $\{\Gamma :$ Ctx $\omega\} \to$ Ty $\Gamma \to$ Ty Γ
next' : Tm Γ $T \to$ Tm Γ $(\triangleright' T)$

Using standard induction for natural numbers, we can also provide an operation corresponding to Löb induction (note that ▷' binds more tightly than ⇒)

löb : $(T :$ Ty $\Gamma) \to$ Tm Γ $(\triangleright' T \Rightarrow T) \to$ Tm Γ T

and show that it produces fixpoints.

löb-is-fixpoint : $(f :$ Tm Γ $(\triangleright' T \Rightarrow T)) \to$
                  löb $T f \cong$ app $f ($next' $($löb $T f))$

The prototypical example in the literature on guarded recursion is the type of guarded streams. We can define in our setting the type

Stream : $\{\Gamma :$ Ctx $\omega\} \to$ Ty Γ

of guarded streams of natural numbers. The intuition is that Stream is isomorphic to the type Nat' ⊠ ▷' Stream, and hence the constructor for streams has the following form.[3]

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[2]Note that in the Agda code the modality ▷' is defined in terms of a more basic modality ▷ that we will not discuss in this abstract. This explains the prime symbol in its name.

[3]This is not completely accurate regarding the difference between ▷ and ▷'. See the full code for details.

str-cons : Tm Γ (Nat' ⊠ (▷' Stream)) → Tm Γ Stream

We can then use Löb induction to define a constant stream of zeros

zeros : Tm ◇ Stream
zeros = löb Stream
            (lam (▷' Stream) $(\iota[\, \ldots \,]$ (str-cons
                 (pair zero' $(\iota[\, \ldots \,]\, \xi)))))$

where ◇ is the empty context and where the … represent two equality proofs for types. More concretely, the first proof has type Stream $[\, \pi \,] \cong$ Stream and this fact is not surprising as Stream is a non-dependent type (the proof itself is also very simple and provided as an operation available to the user). The second proof, on the other hand, is not straightforward although it also boils down to the fact that Stream is a non-dependent type. We consider this an inconvenience in our framework that needs to be dealt with in order to obtain a system that is user-friendly.

4 Related and Future Work

In [9], Jaber et al. describe a translation based on presheaves from extensions of the Calculus of Constructions (CoC) to CoC itself. However, their framework is quite different to work with (there is no CwF-like structure) and they make use of subtypes, which are not present in Agda. Veltri and van der Weide [14] provide presheaf semantics of guarded recursion in Agda in a way which is similar to ours. Both [9] and [14] consider presheaves over a preorder rather than a general category. Guarded recursion can also be presented in Agda using ordered families of equivalences, as in for example https://github.com/metaborg/mj.agda/tree/develop. Again, this approach does not generalize to arbitrary categories. In [2], Bickford formalizes presheaf models for general base categories using the structure of a CwF in Nuprl, which is a proof assistant implementing an extensional version of type theory. To the best of our knowledge, our framework is the first shallow embedding of type theories using presheaves over general categories in intensional type theory. Another approach to formalize presheaf models, is to use Agda's type theory as the internal language of the presheaf topos, like in [13].

As discussed previously, we first want to make our framework more usable by reducing the number of type equality proofs that a user needs to provide. Next, we plan to study different extensions of MLTT using concrete presheaf models with specific base categories. In the first instance, we intend to consider applications involving non-dependent types, but we are planning to add dependent types and a universe type to the framework as well in a later phase. Another interesting path to explore in our framework is multimode type theory as described in [7].

Acknowledgments

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